Lloyd’s method

Definition:

The method to find the least square estimators for location µ and scale ʎ parameters, on the basis of ordered observation is called Lioyed’s method.

Practical Difficulties in application of Lioyed’s Method:

i) By this method we can only estimate scale ʎ and location µ parameters not more than these two.

ii) For large ‘n’ the use of direct method to get the exact solution of minimum variance problem becomes impractical even when “n” small, say n<10. The calculations are time consuming and involve numerical integration.

iii) Application of Lioyeds Method can be used for symmetrical distribution.

Question

Derive the least square estimates of location and scale parameter using order statistic.

OR

Derive the BLUE of the location µ, and scale parameters by least square method using ordered observation.

Answer

In this method we discuss that the BLUE of the location µ and scale ʎ parameter can be obtained by the least square method by using ordered observations $X_{(1)} < X_{(2)} < \cdots X_{(n)}$. $X_{(1)}$ is the smallest ordered statistic and $X_{(n)}$ is the largest ordered statistics. Let make a linear transformation.

$$Y_i = \frac{X_i - \mu}{\lambda}$$

$$\lambda Y_i = X_i - \mu$$

$$X_i = \mu + \lambda Y_i$$

$$E(X_i) = \mu + \lambda \mathrm{E}(Y_i)$$

$$\therefore \mathrm{E}(Y_i) = \alpha_i$$

$$E(X_i) = \mu + \lambda \alpha_i$$

$$\mathrm{E}(X_1) = \mu + \lambda \alpha_1$$

$$\mathrm{E}(X_2) = \mu + \lambda \alpha_2$$

$$\mathrm{E}(X_n) = \mu + \lambda \alpha_n$$

Then

$$\underline{X} = \begin{bmatrix} X1 \\ X2 \\ X3 \\ . \\ . \\ Xn \end{bmatrix}$$

$$\underline{P} = \begin{bmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \\ 1 & \alpha_3 \\ . & . \\ . & . \\ . & . \\ 1_{(n)} & \alpha_{(n)} \end{bmatrix}$$

$$\hat{\theta} = \begin{bmatrix} \hat{\mu} \\ \hat{\lambda} \end{bmatrix}$$

Now

$$Var(X_i) = \mathrm{E}[X_i - \mathrm{E}(Xi)]^2$$

$$Var(X_i) = \mathrm{E}[\mu + \lambda y_i - \mu - \lambda \mathrm{E}(yi)]^2$$

$$Var(X_i) = \lambda^2 \mathrm{E}[yi - \mathrm{E}(yi)]^2$$

$$Var(X_i) = \lambda^2 Var(yi) \qquad \therefore v(yi) = v^n$$

$$= \lambda^2 v^n$$

And

$$\underline{v}^n = \cdot \begin{bmatrix} v_{11} \cdot\cdot v_{12} \cdot\cdot v_{13} \cdots\cdots v_{1n} \\ v_{21} \cdot\cdot v_{22} \cdot\cdot v_{23} \cdots\cdot v_{2n} \\ v_{31} \cdot\cdot v_{32} \cdot\cdot v_{33} \cdots\cdot v_{3n} \\ v_{n1} \cdot\cdot v_{n2} \cdot\cdot v_{n3} \cdots v_{nn} \end{bmatrix}$$

Now

$$\hat{\theta} = \left(\underline{P}'\underline{V}^{-1}\underline{P}\right)^{-1}\left(\underline{P}'\underline{V}^{-1}\underline{X}\right) \qquad \text{(A)}$$

Then θ is a required estimate and variance and covariance of θ is

$$Var - Cov(\hat{\theta}) = (\underline{P}'\underline{V}^{-1}\underline{P})^{-1}\sigma_e^2$$

Now

$$\underline{P} = \begin{bmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \\ 1 & \alpha_3 \\ . & . \\ . & . \\ . & . \\ 1_{(n)} & \alpha_{(n)} \end{bmatrix} \underline{P} = [\underline{I} \quad \underline{\alpha}]$$

$$\underline{P}' = \begin{bmatrix} \underline{I}' \\ \underline{\alpha}' \end{bmatrix}$$

And

$$\left[\underline{P}'\underline{V}^{-1}\underline{P}\right]=\begin{bmatrix}\underline{I}'\\\underline{\alpha}'\end{bmatrix}\underline{v}^{-1}\left[\underline{I}\quad\underline{\alpha}\right]$$

$$=\begin{bmatrix}\underline{I}'\underline{V}^{-1}\underline{I} & \underline{I}'\underline{V}^{-1}\underline{\alpha}\\\underline{\alpha}'\underline{V}^{-1}\underline{I} & \underline{\alpha}'\underline{V}^{-1}\underline{\alpha}\end{bmatrix}$$

$$\left|\underline{P}'\underline{V}^{-1}\underline{P}\right|=\left(\underline{I}'\underline{V}^{-1}\underline{I}\right)\left(\underline{\alpha}'\underline{V}^{-1}\underline{\alpha}\right)-\left(\underline{I}'\underline{V}^{-1}\underline{\alpha}\right)\left(\underline{\alpha}'\underline{V}^{-1}\underline{I}\right)$$

$$=\Delta$$

$$\text{adj}\left(P'V^{-1}P\right)=\begin{bmatrix}\underline{\alpha}'\underline{V}^{-1}\underline{\alpha} & -\underline{I}'\underline{V}^{-1}\underline{\alpha}\\-\underline{\alpha}'\underline{V}^{-1}\underline{I} & \underline{I}'\underline{V}^{-1}\underline{I}\end{bmatrix}$$

$$\left(P'V^{-1}P\right)^{-1}=\frac{adj\left(\underline{P}'\underline{V}^{-1}\underline{P}\right)}{\left|\underline{P}'\underline{V}^{-1}\underline{P}\right|}$$

$$=\begin{bmatrix}\underline{\alpha}'\underline{V}^{-1}\underline{\alpha} & -\underline{I}'\underline{V}^{-1}\underline{\alpha}\\-\underline{\alpha}'\underline{V}^{-1}\underline{I} & \underline{I}'\underline{V}^{-1}\underline{I}\end{bmatrix}/\Delta$$

$$\left(\underline{P}'\underline{V}^{-1}\underline{X}\right)=\begin{bmatrix}\underline{I}'\\\underline{\alpha}'\end{bmatrix}\underline{V}^{-1}\underline{X}$$

$$=\begin{bmatrix}\underline{I}'\underline{V}^{-1}\underline{X}\\\underline{\alpha}'\underline{V}^{-1}\underline{X}\end{bmatrix}$$

Put in eq(A)

$$\hat{\theta}=\begin{bmatrix}\hat{\mu}\\\hat{\lambda}\end{bmatrix}=\left(\underline{P}'\underline{V}^{-1}\underline{P}\right)^{-1}\left(\underline{P}'\underline{V}^{-1}\underline{X}\right)$$

$$=\frac{\begin{bmatrix}\underline{\alpha}'\underline{v}^{-1}\underline{\alpha} & -\underline{I}'\underline{V}^{-1}\underline{\alpha}\\-\underline{\alpha}'\underline{V}^{-1}\underline{I} & \underline{I}'\underline{V}^{-1}\underline{I}\end{bmatrix}\begin{bmatrix}\underline{I}'\underline{V}^{-1}\underline{X}\\\underline{\alpha}'\underline{V}^{-1}\underline{X}\end{bmatrix}}{\Delta}$$

$$\hat{\mu}=\frac{\underline{\alpha}'\underline{V}^{-1}\underline{\alpha}\underline{I}'\underline{V}^{-1}\underline{X}-\underline{I}'\underline{V}^{-1}\underline{\alpha}\underline{\alpha}'\underline{V}^{-1}\underline{X}}{\Delta}$$

$$\hat{\lambda}=\frac{\underline{I}'\underline{V}^{-1}\underline{I}\underline{\alpha}'\underline{V}^{-1}\underline{X}-\underline{\alpha}'\underline{V}^{-1}\underline{I}\underline{I}'\underline{V}^{-1}\underline{X}}{\Delta}$$

Which is required least square estimates for location μ and scale λ parameter.

Now

$$Var - Cov(\underline{\hat{\theta}}) = (\underline{P}'\underline{V}^{-1}\underline{P})^{-1}\sigma_e^2$$

$$= \frac{\begin{bmatrix} \underline{\alpha}'\underline{v}^{-1}\underline{\alpha} & -\underline{I}'\underline{V}^{-1}\underline{\alpha} \\ -\underline{\alpha}'\underline{V}^{-1}\underline{I} & \underline{I}'\underline{V}^{-1}\underline{I} \end{bmatrix}\sigma_e^2}{\Delta}$$

So

$$\mathrm{var}(\hat{\mu}) = \frac{\underline{\alpha}'\underline{v}^{-1}\underline{\alpha}\sigma_e^2}{\Delta}$$

$$\therefore \lambda = \sigma_e^2$$

$$\mathrm{var}(\hat{\lambda}) = \frac{\underline{I}'\underline{V}^{-1}\underline{I}\lambda^2}{\Delta}$$

$$Cov(\hat{\mu}, \hat{\lambda}) = \frac{-\underline{\alpha}'\underline{v}^{-1}\underline{I}\sigma_e^2}{\Delta}$$

OR

$$= \frac{-\underline{I}'\underline{V}^{-1}\underline{\alpha}\sigma_e^2}{\Delta}$$

PRACTICAL:

Let ‘X’ be a random variable with probability density function.

If 1.2315 < 2.0897 are the variable ordered observation from the two parameter of Rayleigh distribution. Obtain the best linear unbiased estimator of parameter $'\mu' \, and \, '\lambda'$. Using the Lioyd’s method while it is given below:

$$\underline{\alpha} = [o.5116634 \quad 0.8566445 \quad 1.2903729]$$

And

$$\underline{v} = \begin{bmatrix} 0.0715340 & 0.478340 & 0.0335480 \\ 0.0473340 & 0.09948350 & 0.070878 \\ 0.0335480 & 0.0708780 & 0.1618271 \end{bmatrix}$$

Where,

$$\underline{\hat{\theta}} = \begin{bmatrix} \hat{\mu} \\ \hat{\lambda} \end{bmatrix} = \left(\underline{P'}\underline{Y}^{-1}\underline{P}\right)^{-1}\left(\underline{P'}\underline{V}^{-1}\underline{X}\right)$$

SOLUTION:

It is given that:

$$\underline{X} = \begin{bmatrix} 1.2315 \\ 2.0897 \\ 2.9513 \end{bmatrix}$$

$$\underline{\alpha} = [o.5116634 \quad 0.8566445 \quad 1.2903729]$$

And

$$\underline{v} = \begin{bmatrix} 0.0715340 & 0.478340 & 0.0335480 \\ 0.0473340 & 0.09948350 & 0.070878 \\ 0.0335480 & 0.0708780 & 0.1618271 \end{bmatrix}$$

As the Rayleigh dist has two parameter. So, we estimate these two parameter using Lioyd’s method.

$$\underline{\hat{\theta}} = \begin{bmatrix} \hat{\mu} \\ \hat{\lambda} \end{bmatrix} = \left(\underline{P}'\underline{Y}^{-1}\underline{P}\right)^{-1}\left(\underline{P}'\underline{V}^{-1}\underline{X}\right)$$

Where,

$$P = \begin{bmatrix} 1 & 0.5116634 \\ 1 & 0.8566445 \\ 1 & 1.2903729 \end{bmatrix}$$

$$P' = \begin{bmatrix} 1 & 1 & 1 \\ 0.5116634 & 0.8566445 & 1.2903729 \end{bmatrix}$$

As we know that:

$$\underline{v}^{-1} = \frac{adj\underline{v}}{|\underline{v}|}$$

The co-factor are:

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 0.099483 & 0.070878 \\ 0.0478340 & 0.1610271 \end{vmatrix}$$

= 0.01609913 – 0.00502369 = 0.01107544

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0.0478340 & 0.070878 \\ 0.0478340 & 0.1610271 \end{vmatrix}$$

= - (0.00774084 – 0.00237780) = - 0.0536302

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 0.0478340 & 0.0994835 \\ 0.0335480 & 0.070878 \end{vmatrix}$$

= 0.00339038 – 0.00333747 = 0.00005191

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 0.0478340 & 0.335480 \\ 0.090878 & 0.1618771 \end{vmatrix}$$

= - (0.00774084 – 0.002377820) = - 0.00536302

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 0.0715340 & 0.335480 \\ 0.335480 & 0.1618771 \end{vmatrix}$$

= (0.01157614 – 0.00112547) = 0.01045067

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 0.0715340 & 0.0478340 \\ 0.335480 & 0.070878 \end{vmatrix}$$

= - (0.00507018 – 0.00160774) = - 0.00346545

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 0.0715340 & 0.0478340 \\ 0.0478340 & 0.0994835 \end{vmatrix}$$

= 0.00711645 – 0.00228809 = 0.00482856

$$|\underline{V}| = V_{11}A_{11} + V_{12}A_{12} + V_{13}A_{13}$$

= 0.7115340 (0.01099577) + 0.0478340 (0.00536302)+0.0335480(0.0

$|\underline{V}|$ = 0.0005318

$$adj(\underline{v}) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 0.01107544 & -0.00536302 & 0.0005291 \\ -0.00536302 & 0.01045607 & -0.000346545 \\ 0.0005291 & 0.003465456 & 0.00482856 \end{bmatrix}$$

$$v^{-1} = \frac{adjv}{|v|}$$

$$= \frac{1}{0.00053181} \begin{bmatrix} 0.01107544 & -0.00536302 & 0.0005291 \\ -0.00536302 & 0.01045607 & -0.000346545 \\ 0.0005291 & 0.003465456 & 0.00482856 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} 20.52533409 & -10.0844626 & 0.09949041 \\ -10.0844626 & 19.65113480 & -6.51633102 \\ 0.09943041 & -0.5153102 & 9.07710720 \end{bmatrix}$$

Now,

$$\underline{P}^{-1}\underline{V}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0.5116634 & 0.08566445 & 1.2907729 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} 20.52533409 & -10.0844626 & 0.09949041 \\ -10.0844626 & 19.65113480 & -6.51633102 \\ 0.09943041 & -0.5153102 & 9.07710720 \end{bmatrix}$$

$$= \begin{bmatrix} 10.69114908 & 3.05033752 & 2.66226659 \\ 2.14544541 & 3.6563730 & 6.18416036 \end{bmatrix}$$

$$(\underline{P}'\underline{V}^{-1}\underline{P}) = \begin{bmatrix} 10.69114908 & 3.05033752 & 2.66226659 \\ 2.14544541 & 3.6563730 & 6.18416036 \end{bmatrix} \begin{bmatrix} 1 & 0.5116634 \\ 1 & 0.8566445 \\ 1 & 1.2903729 \end{bmatrix}$$

$$= \begin{bmatrix} 16.40375319 & 11.51864121 \\ 11.51864121 & 11.83593194 \end{bmatrix}$$

$$\left|\underline{P}'\underline{V}^{-1}\underline{P}\right| = (16.40375319 \times 11.83593794)\text{-}(11.586412 \times 11.51846121)$$

$$= 61.4461099$$

$$adj(\underline{P}'\underline{V}^{-1}\underline{P}) = \begin{bmatrix} 11.83593194 & -11.51864121 \\ -11.51864121 & 16.40375319 \end{bmatrix}$$

$$\left(\underline{P}'\underline{V}^{-1}\underline{P}\right)^{-1} = \frac{adj\left(\underline{P}'\underline{V}^{-1}\underline{P}\right)}{\det\left|\underline{P}'\underline{V}^{-1}\underline{P}\right|}$$

$$= \frac{1}{61.4461099} \begin{bmatrix} 11.83593194 & -11.51864121 \\ -11.51864121 & 16.40375319 \end{bmatrix}$$

$$= \begin{bmatrix} 0.19253366 & -0.1873233 \\ -0.1873233 & 16.40375319 \end{bmatrix} \qquad \text{(B)}$$

$$\left(\underline{P}'\underline{V}^{-1}\underline{P}\right) = \begin{bmatrix} 10.69114908 & 3.05033752 & 2.66226659 \\ 2.14544541 & 3.6563730 & 6.18416036 \end{bmatrix} \begin{bmatrix} 01.2315 \\ 2.0837 \\ 2.09513 \end{bmatrix}$$

$$=\begin{bmatrix} 10.69114908(1.2315)+3.05033752(2.0837)+2.66226659(2.09513) \\ 2.14544541(1.2315)+3.26563730(2.0837)+6.18416030(2.09513) \end{bmatrix}$$

$$=\begin{bmatrix} 27.397587791 \\ 27.6233848 \end{bmatrix}$$

Put ‘B’ and ‘C’ in eq (A)

$$\hat{\theta}=\begin{bmatrix} \hat{\mu} \\ \hat{\lambda} \end{bmatrix}=\begin{bmatrix} 0.19253366 & -0.1873233 \\ -0.1873233 & 16.40375319 \end{bmatrix}\begin{bmatrix} 27.397587791 \\ 27.6233848 \end{bmatrix}$$

$$\begin{bmatrix} \hat{\mu} \\ \hat{\lambda} \end{bmatrix}=\begin{bmatrix} 0.09910856 \\ 2.23740239 \end{bmatrix}$$

So, $\hat{\mu}$ =0.09910856 is the best linear unbiased estimate of location parameter $\hat{\mu}$ and $\hat{\lambda}$ =2.23740239 is the best linear unbiased estimate scale parameter $\hat{\lambda}$ which are obtain by Lioyd’s method.